Worcester County Mathematics League
Varsity Meet 1 - October 28, 2020
Round 1 - Arithmetic

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Evaluate the expression
$$\frac{32 \div 5^0 - 6 \times 8 - 4 \times 8}{6 + 6 \div 3 - 4 + 11 \times 4}$$

2. Let $a \bigtriangledown b = a^b - b$ and $a \bigtriangleup b = a^b - a$. Evaluate:
$$[((2 \bigtriangledown 0) \bigtriangledown 2) \bigtriangledown 0] - [((2 \bigtriangleup 0) \bigtriangleup 2) \bigtriangleup 0]$$

3. Let $F(a,b,c) = \frac{a+b}{c^2}$, $G(a,b,c) = \frac{c^2}{a+b}$. Evaluate $F(G(-173, 614, 63), G(-333, 622, 68), F(-173, 614, 63)))$

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. ____________________ Tantasqua, Quaboag, QSC

Worcester County Mathematics League
Varsity Meet 1 - October 28, 2020
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Solve for x if x must be a positive integer:

$$x^2 - 70 = 3x$$

2. A rock climbing gym charges nonmembers \$16 per day to use the gym and \$8 for equipment rental. Members pay a yearly fee of \$950 for unlimited gym usage and \$6 per day for equipment rental. How many days must a member use the gym so that his total payment (yearly fee plus per use fee) is less than it would be if he were a nonmember? (The member rents equipment every day he uses the gym).

3. The sum of four numbers l, m, n, and p is 771. The numbers have ratios $(l : m) = (2 : 3)$, $(m : n) = (5 : 4)$, $(n : p) = (5 : 6)$. Express the numbers as an ordered 4-tuple: (l,m,n,p)

ANSWERS

(1 pt) 1. $x \epsilon \{$ ________________ $\}$

(2 pts) 2. ________________

(3 pts) 3. $(l, m, n, p) = ($ ____________ $)$

Tantasqua, Hopkinton, Auburn

Worcester County Mathematics League

Varsity Meet 1 - October 28, 2020

Round 3 - Set Theory

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Ms. Mathy asked each student in her class how many brothers and sisters the student had. Five students had no brothers and no sisters, 12 students had at least one brother, 15 students had at least one sister, and 6 students had at least one brother and one sister. How many students were in Ms. Mathy's class?

2. Sets A and B are listed below. Set C has 15 subsets, not including itself. If $(A \cup B) \cap C$ has two elements, how many elements are in $A \cup B \cup C$?

$$\begin{aligned} A &= \{1, 2, 3, 4, 5, 6\} \\ B &= \{4, 5, 6, 7, 8, 9, 10, 11, 12\} \end{aligned}$$

3. For each odd positive integer n less than 100, let S_n be the set of positive multiples of n less than 100. Find the largest possible size (number of sets) of a collection of sets $\{S_n\}$ that are pairwise disjoint. That is, any two sets in the collection are disjoint.

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Algonquin?, Shrewsbury, QSC

Worcester County Mathematics League

Varsity Meet 1 - October 28, 2020

Round 4 - Measurement

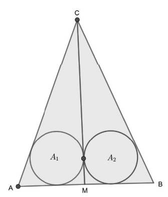
All answers must be in simplest exact form in the answer section.

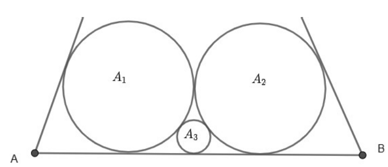
NO CALCULATORS ALLOWED

1. If the largest possible sphere is placed in a rectangular box of dimensions 6cm by 7cm by 9cm, the volume of the space in the box that is outside of the sphere can be expressed as $(m - n\pi)\ cm^3$. Find the ordered pair (m, n).

2. In square $ABCD$, M is the midpoint of $\overline{AB}$ and N is located on $\overline{CD}$ such that $\frac{CN}{ND} = \frac{1}{3}$. If $MN = 2\sqrt{17}$, find the perimeter of $ABCD$.

3. Isosceles $\triangle ABC$ has altitude $\overline{CM}$, a base $\overline{AB}$ of length 10 and legs of length 13. Circles with areas A_1 and A_2 are inscribed in $\triangle AMC$ and $\triangle BMC$ as shown at left below. A third circle is tangent to base $\overline{AB}$ as well to both of the first two circles, as shown in the zoomed-in figure to the right. The sum of the three areas $A_1 + A_2 + A_3$ can be expressed as a $k\pi$. Find k and express it as a fraction $\frac{a}{b}$ in lowest terms.





ANSWERS

(1 pt) 1. (______________________)

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Quabog, St. John's, QSC

Worcester County Mathematics League
Varsity Meet 1 - October 28, 2020
Round 5 - Polynomial Equations

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find the remainder when polynomial $p(x) = x^{99} + 14$ is divided by $x + 1$.

2. Given $a = b + 5$, what values of a and b will result in the smallest possible value of their product ab? Express your answer as an ordered pair (a, b), with a and b expressed as fractions in lowest terms.

3. Given cubic polynomial $f(x)$, where $f(1) = 6$, $f(2) = 11$, $f(3) = 24$, and $f(4) = 51$, find $f(5)$.

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. ____________________ Hopkinton, Tahanto, AMSA

Worcester County Mathematics League
Varsity Meet 1 - October 28, 2020
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Evaluate the following expression and express your answer as an improper fraction in lowest terms.
$$(((2+0)^{-1}+2)^{-1}+0)^{-1}-(((2+0)^{-1}+2)^{-1}+1)^{-1}$$

2. Find all sets of integers $\{x, y\}$, not necessarily distinct or positive, where
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{5}$$

3. Let $A=\{x : \left|x^2-9\right| \leq 5\}$ and $B=\{x : \left|\sqrt{}(x-2\right| > \frac{1}{2}\}$. Express $A^c \cap B$ using interval notation, where A^c is the complement of A.

4. A right cylindrical oil tank is 15 feet tall and its circular bases each have a diameter of 4 feet. When the tank is lying on level ground along the length of its curved surface, the oil inside the tank is 3 feet deep at the deepest point. The tank is then tipped up onto a base without losing any of the oil. The height of the oil can be expressed as $a+\frac{b\sqrt{(3)}}{c\pi}$, where a, b, and c are integers and b and c are relatively prime. Find the ordered triple (a, b, c).

5. A lawn is 90 feet by 120 feet. How wide must a strip of uniform width be cut around the outside of the lawn so that half of the area of the lawn is mowed and half is uncut?

6. Evaluate the following expression:
$$-3^2-(-3)^2-4^2+|-2-2|$$

7. An airport parking garage offers free parking for the first hour and changes a flat rate of \$15.00 if a car is parked less than four hours or more than an hour. It charges \$3.75 per hour or any part of an hour above four hours. How much does the garage charge for a car that is parked for 8 hours and 15 minutes?

8. There are 930 students at Hilbert High School, and each student takes at least one class of Math, Biology or History. The number of student taking Math equals the number of students taking Biology and the number of students taking History. Likewise, the number of students taking Math and Biology equals the number of students taking Biology and History and the number taking Math and History. The total number of students taking Math is equal to the total number of students not taking Math. If 180 students take all three subjects, how many students take Math and History?

9. Find all real solutions of:
$$|x-3| \cdot |x+4|=8$$

Worcester County Mathematics League
Varsity Meet 1 - October 28, 2020
Team Round Answer Sheet

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

5. ______________________________ feet

6. ______________________________

7. ______________________________

8. ______________________________

9. ______________________________

QSC, Bancroft, AMSA, AMSA, Notre Dame, Tahanto, Auburn, Worcester Academy, St. John's

Worcester County Mathematics League
Varsity Meet 1 - October 28, 2020
Answer Key

Round 1 - Arithmetic

1. -1
2. 2
3. 2025

Round 2 - Algebra I

1. $x = 10$ (the answer "10" also accepted)
2. 53
3. $(l, m, n, p) = (150, 225, 180, 216)$

Round 3 - Set Theory

1. 26
2. 14
3. 33

Round 4 - Measurement

1. $(378, 36)$
2. 32
3. $\frac{33}{4}$

Round 5 - Polynomial Equations

1. 13
2. $(\frac{5}{2}, \frac{-5}{2})$
3. 98

Team Round

1. $\frac{25}{14}$
2. $\{6, 30\}, \{4, -20\}, \{10, 10\}$ (need all three unordered pairs)
3. $[0, 2) \cup (\frac{25}{4}, \inf)$
4. $(10, 15, 4)$
5. 15
6. -30
7. \$33.75
8. 215
9. $\{-5, 4, \frac{-1 \pm \sqrt{17}}{2}\}$

Round 1 solutions

1. Applying parentheses to the expression to make the order of operations more explicit:

$$\frac{(32\div(5^0))-(6\times 8)-(4\times 8)}{6+(6\div 3)-4+(11\times 4)} = \frac{(32\div 1)-48-32}{6+2-4+44}$$

$$= \frac{32-48-32}{8+40}$$

$$= \frac{-48}{48} = -1$$

2. $$(((2\nabla 0)\nabla 2)\nabla 0) = ((2^0-0)\nabla 2)\nabla 0 = (1\nabla 2)\nabla 0 = (1^2-2)\nabla 0 = (-1)\nabla 0 = (-1)^0-0$$
$$=1$$

$$(((2\Delta 0)\Delta 2)\Delta 0) = ((2^0-2)\Delta 2)\Delta 0 = ((-1)\Delta 2)\Delta 0 = ((-1)^2-(-1))\Delta 0 = (1+1)\Delta 0 = 2\Delta 0$$
$$= 2^0-2 = 1-2 = -1$$

$$(((2\nabla 0)\nabla 2)\nabla 0) - (((2\Delta 0)\Delta 2)\Delta 0) = 1-(-1) = 1+1 = 2$$

3. (1^*) $$G(-173, 614, 63) = \frac{63^2}{-173+614} = \frac{63^2}{441} = \frac{63^2}{21^2} = \left(\frac{63}{21}\right)^2 = 3^2 = 9$$

(2^*) $$F(-173, 614, 63) = \frac{1}{G(-173,614,63)} = \frac{1}{9}$$

(3^*) $$G(-333, 622, 68) = \frac{68^2}{-333+622} = \frac{68^2}{289} = \frac{68^2}{17^2} = \left(\frac{68}{17}\right)^2 = 4^2 = 16$$

$$F((1^*),(3^*),(2^*)) = F(9, 16, 1/9) = \frac{9+16}{(1/9)^2} = \frac{25}{1/81} = 81\cdot 25 = 2025$$

1. -1

2. 2

3. 2025

Round 2

1. Solve: $x^2 - 70 = 3x$, $x^2 - 3x - 70 = 0$ Factor: $-10 \cdot 7 = -70, -10+7=-3$

$(x-10)(x+7) = 0 \Rightarrow x \in \{-7, 10\}$

2. Let d = number of days that the climber uses the gym.
Let $C(d)$ = climber's cost with annual pass, for d days of gym use
$N(d)$ = climber's cost without annual pass " " " " " "

Then $C(d) = 950 + 6d$; $N(d) = 18d + 6d = 24d$.
Find d such that $C(d) < N(d)$; $950 + 6d < 24d$; $950 < 18d$; $d > \frac{950}{18}$
$d > \frac{475}{9} \approx 52\frac{7}{9} \Rightarrow d = 53$ is the minimum number of days of gym use for the annual pass to save $.

3. $l + m + n + p = 771$
$(l:m) = (2:3) \Rightarrow (l:m) = (2r:3r)$
$(m:n) = (5:4) \Rightarrow (m:n) = (5s:4s)$
$3r = 5s \Rightarrow$ let $r = 5u$, $s = 3u$; $(l:m:n) = (10u:15u:12u)$

$(l:m:n) = (10u:15u:12u)$
$(n:p) = (5v:6v)$
$12u = 5v$ let $u = 5w$, $v = 12w$; $(l:m:n:p) = (50w:75w:60w:72w)$

$l + m + n + p = 50w + 75w + 60w + 72w = 257w = 771 \Rightarrow w = \frac{771}{257} = 3$

$(l, m, n, p) = (150, 225, 180, 216)$

Check: $(150 : 225) = 2:3$
$(225 : 180) = 5:4$
$(180 : 216) = 5:6$
$150 + 225 + 180 + 216 = 771$

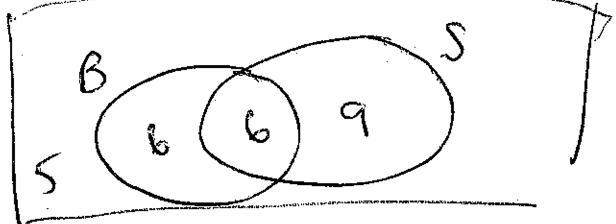
1. $x \in$ $\{-7, 10\}$

2. 53 days

3. $(150, 225, 180, 216)$

Round 3

1.



1. $\underline{26}$

2. $\underline{14}$

3. $\underline{33}$

$B = \{$students with at least one brother$\}$
$S = \{$students with at least one sister$\}$
$\Omega = \{$students in Ms. Mathy's class$\}$

$|B| = |B \cap S| + |B \cap S^c|$, where $|S| =$ # of elements in set S.

$12 = 6 + |B \cap S^c| \Rightarrow |B \cap S^c| = 6$

$|S| = |B \cap S| + |B^c \cap S|$

$15 = 6 + |B^c \cap S| \Rightarrow |B^c \cap S| = 9$

$|\Omega| = 5 + 6 + 6 + 9 = 26$

2. Set C has $15 + 1 = 16$ subsets including itself.
If $|C| = n$, then C has 2^n subsets, so $2^n = 16$, $n = 4$.

Note that the elements in A and B are the integers from 1 to 12,
so $A \cup B = \{1, 2, 3, \ldots, 12\}$ and $|A \cup B| = 12$.
Let $A \cup B = D$. Then $|D \cap C| = 2$ and $|D| = 12$

By the principle of inclusion-exclusion;
$|A \cup B \cup C| = |D \cup C| = |D| + |C| - |D \cap C| = 12 + 4 - 2 = 14$

3. see next page

Round 3, problem 3

Two sets are disjoint if they share no elements Therefore if sets S_i and S_j share an element, at most one of those two sets can be in the collection.

Consider all odd integers k for which S_k has one element. Then $k \in \{51, 53, \ldots, 99\}$ and there are 25 sets with a single element.

Consider all odd integers k for which S_k has exactly two elements. Then $k \in \{35, 37, \ldots, 49\}$, and $S_k = \{k, 2k\}$. Note that all these sets are disjoint from the single element sets because $2k$ is even, and there are 8 of these sets.

Consider odd integers k between 13 and 33. In each case, $3k \in S_k$, and $3k$ is an element of the ~~se~~ collections with one or two elements, and the number of sets cannot be increased by adding any one of these sets. Likewise, for $k = 7, 9,$ or 11, $5k \in S_k$, and $5k$ is shared with at least one set in the collection. A similar statement can be made for $k = 1, 3,$ or 5. Thus largest size is $8 + 25 = 33$ sets.

Round 4

1. $(378, 36)$

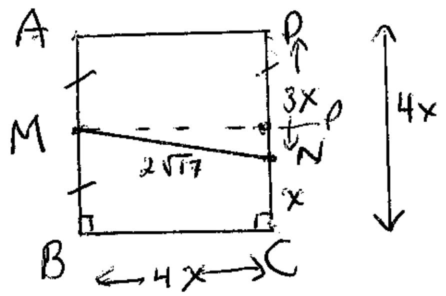
2. 32

3. $\frac{33}{4}$

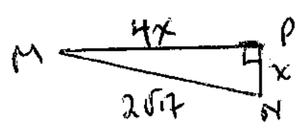
1. Let V_B = Volume of the box
V_S = Volume of the sphere
V_x = Volume of the remaining space
$= V_B - V_S$

The largest possible sphere that fits in the box has a diameter equal to the smallest dimension of the box, or 6cm. The radius would then be $\frac{6\text{cm}}{2} = 3\text{cm}$.

$$V_B = 6(7\cdot 9) = 6(63) = 378$$
$$V_S = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3)^3 = 4\cdot 3^2\pi = 36\pi$$
$$V_x = V_B - V_S = 378 - 36\pi \qquad (m,n) = (378, 36)$$

2. 

Draw the figure, at left. Call P the midpoint of $\overline{DC}$. Let $CN = x$. Then $NO = 3x$, and CP



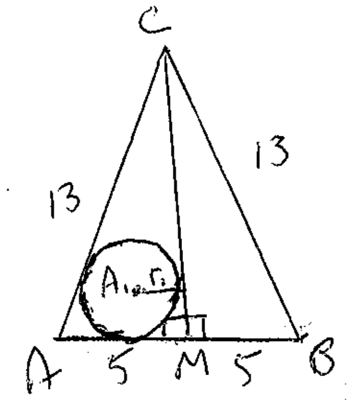
Draw $\triangle MPN$, which is a right $\triangle$. By the Pythagorean Theorem
$$x^2 + (4x)^2 = (2\sqrt{17})^2 = 4\cdot 17$$
$$x^2 + 16x^2 = 17x^2 = 4\cdot 17 \Rightarrow x^2 = 4$$
$$x = 2.$$

The perimeter of ABCD $= 4x + 4x + 4x + 4x$
$= 16x = 16(2) = 32$

3. see next page

Round 4, problem 3

3. First, find the radii of the first two circles to find A_1 and A_2.



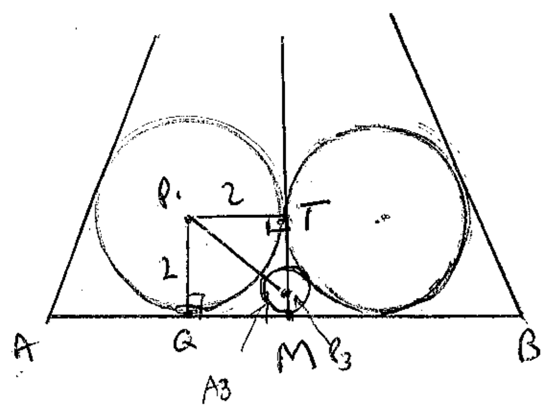
The altitude $\overline{CM}$ divides $\triangle ABC$ into congruent right triangles $\triangle AMC$ and $\triangle BMC$. There fore the two inscribed circles are congruent and $A_1 = A_2$. Note: the radius of the circles $= r_1$
The area of a triangle is related to the radius of its inscribed circle as follows:

$A_1 = A_{AMC} = \frac{1}{2} p r_1$, where p is the perimeter of $\triangle AMC$, r_1 is radius.

Now $CM^2 + AM^2 = AC^2$, or $CM^2 = 13^2 - 5^2 = 169 - 25 = 144$; $CM = 12$.

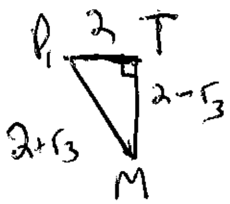
$p = 5 + 12 + 13 = 30$; $A_1 = \frac{1}{2}(30) r_1 = 15r = \frac{1}{2}(AM)(CM) = \frac{1}{2} 5 \cdot 12 = 30$

$15r = 30 \Rightarrow \boxed{r_1 = 2}$ $\quad A_1 = A_2 = \pi r^2 = \pi 2^2 = 4\pi$,



Draw an expanded view of the base of $\triangle ABC$. Let P_1 be the center of one inscribed circle, Q be the point of tangency with $\overline{AB}$, T be the point on $\overline{CM}$ that is the point of tangency for the two inscribed circles, P_1 be the center of the circle. Then P_1QMT is a square with side length 2.

Draw the radii for circles P_1 and P_3 to the point of tangency between the circles. Then P_1P_3 is equal to $r_1 + r_3$, the radii of $\odot P_1$ and $\odot P_3$. Because $MT = 2$, $P_1T = 2$, and $MP_3 = r_3$, The length $TM = 2 - r_3$. Writing the Pythagorean Theorem:



$$2^2 + (2 - r_3)^2 = (2 + r_3)^2$$

$$4 + 4 - 4r_3 + r_3^2 = 4 + 4r_3 + r_3^2,$$

or

$$4 = 8r_3$$

$$\boxed{r_3 = \frac{1}{2}}$$

$A_3 = \pi r^2 = \pi (1/2)^2 = \frac{\pi}{4}$

$A_1 + A_2 + A_3 = 4\pi + 4\pi + \pi/4$

$= (8 + \frac{1}{4})\pi$

$= \boxed{\frac{33}{4}\pi}$

Round 5

1. $\underline{13}$
2. $\underline{-\frac{25}{4}}$
3. $\underline{98}$

1. Applying the remainder theorem:
$P(x) = Q(x)(x+1) + R$; $R = P(-1)$.
$P(-1) = (-1)^{99} + 14 = -1 + 14 = 13$

2. $a = b+5$, $ab = b(b+5) = b^2 + 5b$.
Complete the square: $ab = b^2 + 5b + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2$
$= \left(b + \frac{5}{2}\right)^2 - \frac{25}{4}$

Note that $\left(b + \frac{5}{2}\right)^2 \geq 0$, so ab is minimized when $b = -\frac{5}{2}$, or $ab \geq -\frac{25}{4}$

3. Using the method of differences, let
$Q(x) = f(x+1) - f(x)$ (first difference)
$R(x) = Q(x+1) - Q(x)$ (second difference)
$S(x) = R(x+1) - R(x)$ (third difference)

Note that the third difference will be a constant for a cubic polynomial.
Calculate $Q(1)$, $Q(2)$, $Q(3)$ from given $f(x)$ values.

	$x=1$	$x=2$	$x=3$	$x=4$
$f(x)$	6	11	24	51
$Q(x)$	$11-6=5$	$24-11=13$	$51-24=27$	
$R(x)$	$13-5=8$	$27-13=14$		
$S(x)$	$14-8=6$.			

Now: $S(x) = 6 = R(x+1) - R(x) = R(3) - R(2) = R(3) - 14 \Rightarrow R(3) = 14 + 6 = 20$
Likewise, $Q(4) = Q(3) + R(3) = 27 + 20 = 47$
$f(5) = f(4) + Q(4) = 51 + 47 = 98$
$\boxed{f(5) = 98}$

An alternate method is to set $f(x) = ax^3 + bx^2 + cx + d$, write a system of four linear equations using $f(1), f(2), f(3), f(4)$, and solve for (a, b, c, d)

Team Round Solutions

Wocomal Varsity Meet 1
2020-2021
Oct. 28, 2020

① Let $(((2+0)^{-1}+2)^{-1}+0)^{-1} - (((2+0)^{-1}+2)^{-1}+1)^{-1} = A - B$

Then $(2+0)^{-1} = \frac{1}{2}$, $((2+0)^{-1}+2)^{-1} = (\frac{1}{2}+2)^{-1} = (\frac{5}{2})^{-1} = \frac{2}{5}$ and:

$A = (((2+0)^{-1}+2)^{-1}+0)^{-1} = (\frac{2}{5}+0)^{-1} = (\frac{2}{5})^{-1} = \frac{5}{2}$

$B = (((2+0)^{-1}+2)^{-1}+1)^{-1} = (\frac{2}{5}+1)^{-1} = (\frac{7}{5})^{-1} = \frac{5}{7}$

$A - B = \frac{5}{2} - \frac{5}{7} = 5(\frac{1}{2} - \frac{1}{7}) = 5(\frac{7-2}{14}) = 5\frac{5}{14} = \boxed{\frac{25}{14}}$

---

② Multiply both sides of $\frac{1}{x} + \frac{1}{y} = \frac{1}{5}$ by $5xy$: $(x \neq 0, y \neq 0)$

$5y + 5x = xy$ or $xy - 5x - 5y = 0$

Add 25 to both sides: $xy - 5x - 5y + 25 = 25$

Factor: $(x-5)(y-5) = 25$

Note that x and y are integers, so $(x-5)$ and $(y-5)$ are integers.
There are possibilities: $\{(x-5),(y-5)\} = \{5,5\}, \{-5,-5\}, \{1,25\}, \{-1,-25\}$

$\{(x-5),(y-5)\} = \{5,5\} \Rightarrow \{x,y\} = \{10,10\}$

$\{(x-5),(y-5)\} = \{-5,-5\} \Rightarrow \cancel{\{x,y\} = \{0,0\}} \Rightarrow$ not possible because $(\frac{1}{0})$ is undefined in orig. eqn.

$\{(x-5),(y-5)\} = \{1,25\} \Rightarrow \{x,y\} = \{6,30\}$

$\{(x-5),(y-5)\} = \{-1,-25\} \Rightarrow \{x,y\} = \{4,-20\}$

There are three unordered pairs in total: $\{10,10\}, \{6,30\}, \{4,-20\}$

Each pair can be checked: $\frac{1}{10} + \frac{1}{10} = \frac{1}{5}$; $\frac{1}{6} + \frac{1}{30} = \frac{1}{5}$; $\frac{1}{4} - \frac{1}{20} = \frac{1}{5}$

3. $A = \{x : |x^2 - 9| \le 5\}$ $B = \{x : |\sqrt{x} - 2| > \tfrac{1}{2}\}$

A: $-5 \le x^2 - 9 \le 5$

$4 \le x^2 \le 14$

$2 \le x \le \sqrt{14}$

or $-\sqrt{14} \le x \le -2$

A^c: $-2 < x < 2$ or $x > \sqrt{14}$ or $x < -\sqrt{14}$

B. $\begin{cases} \sqrt{x} - 2 > 1/2 & \text{or} \\ \sqrt{x} - 2 < -1/2 \end{cases}$

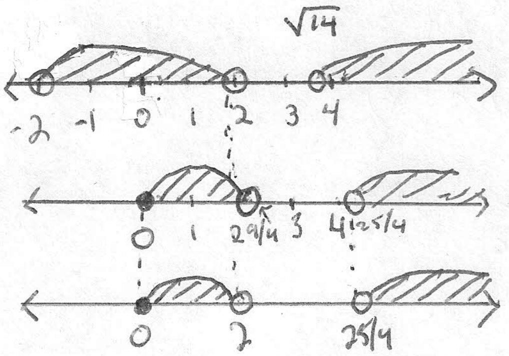
$\begin{cases} \sqrt{x} > 5/2 & \text{or} \\ \sqrt{x} < 3/2 \end{cases}$ Add 2

$\begin{cases} x > 25/4 & \text{or} \\ x < 9/4 \end{cases}$ Square both sides

and $x \ge 0$

Graph A^c, B and $A^c \cap B$

($x \le -\sqrt{14}$ solutions not shown) A^c



(common to A^c and B)

$[0, 2) \cup (25/4, \infty)$

Express the solution using interval notation

$$\boxed{[0, 2) \cup (\tfrac{25}{4}, \infty)}$$

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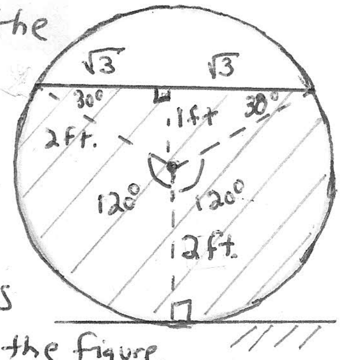
4. Let d = depth of the oil when the tank is upright (in feet). The ratio of the depth of the oil to the height of the tank $(\frac{d}{15})$ (when upright) is equal to the fraction of the area of the base that is filled with oil when the tank is on its side (x) because the volume of oil does not change when the tank is tipped. The circular base of the tank is shown in the figure, where the radius of the circle is 2 ft. The area of the base filled with oil can be divide into a sector and an isosceles triangle with leg lengths 2 ft (the radius). The 3 ft. oil depth extends one foot past the circle center, intersecting the base of the isosceles triangle, and splitting it into congruent right triangles.



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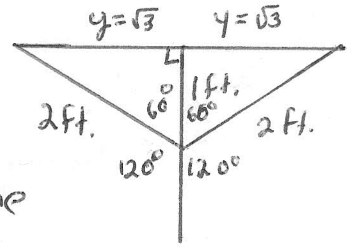
4. (cont.) The right triangles have side lengths $(1, y, 2)$, and $y=\sqrt{3}$ by the Pythagorean Theorem $(1^2+y^2=2^2)$. Therefore the length of the base is $2\sqrt{3}$ and the area of the isosceles triangle is $\frac{1}{2}(2\sqrt{3}\cdot 1)$, and is equal to $\sqrt{3}$ ft.



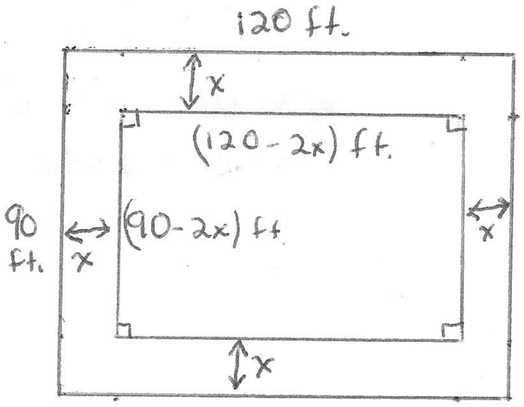
The right triangle is a $30°$-$60°$-$90°$ triangle, as identified by the $(1:\sqrt{3}:2)$ ratio of its side lengths. The measure of the sector's arc is therefore $360°-60°-60°=240°$, and the area of the sector is $\frac{240°}{360°}\pi(2\text{ ft})^2 = \frac{2}{3}4\pi\text{ ft}^2 = \frac{8}{3}\pi\text{ ft}^2$

Thus, the area of the base filled with oil is $(\sqrt{3}+\frac{8}{3}\pi)\text{ft}^2$.

The area of the base is $\pi r^2 = \pi(2\text{ ft})^2 = 4\pi\text{ ft}^2$, and the fraction of the base filled with oil $x = \frac{\sqrt{3}+\frac{8}{3}\pi}{4\pi}$. Returning to the initial statement, $\frac{d}{15}=x$, or $d=15x$, and the depth of the oil $d = 15\left(\frac{\sqrt{3}+\frac{8}{3}\pi}{4\pi}\right) = \frac{15\sqrt{3}+40\pi}{4\pi} = 10+\frac{15\sqrt{3}}{4\pi}$

Then $a+\frac{b\sqrt{3}}{c\pi} = 10+\frac{15\sqrt{3}}{4\pi} = \boxed{(10, 15, 4)}$

5. First, draw a rectangle and label dimensions 90×120 (feet.) Let x be the width of the strip of grass that is mowed (in feet). The unmowed area of grass is a rectangle of dimensions $(90-2x)\times(120-2x)$. Set that area equal to half the area of the full lawn, and solve for x:



$(90-2x)(120-2x) = \frac{1}{2}90(120)$; $90(120) - 2x(90+120) + (2x)^2 = 45(120)$

$4x^2 - 420x + 45(120) = 0$; (divide by 4); $x^2 - 105x + 30\cdot 45 = 0$; $30\cdot 45 = 90\cdot 15$

Factor: $(x-15)(x-90)=0$; $x = 15$ ft or $x=90$; discard $x=90 \Rightarrow \boxed{x=15\text{ ft}}$

Team Round Solutions

Wocomal Varsity Meet I
Oct. 28, 2020

6, $-3^2 - (-3)^2 - 4^2 + |-2-2| = -3^2 - 3^2 - 4^2 + |-4|$

$= -9 - 9 - 16 + 4 = -18 - 16 + 4$

$= \boxed{-30}$

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7. The car is parked (8 hrs 15 min − 4 hrs) = 4 hrs 15 min longer than four hours. Rounding up, the hourly charge is applied to 5 hours, and that charge is added to the flat rate of \$15. The total charge for 8 hrs. 15 min is:

$\$15.00 + 5(\$3.75) = \$15 + \$18.75 = \boxed{\$33.75}$

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8. Let $M = \{\text{Students taking Math}\}$

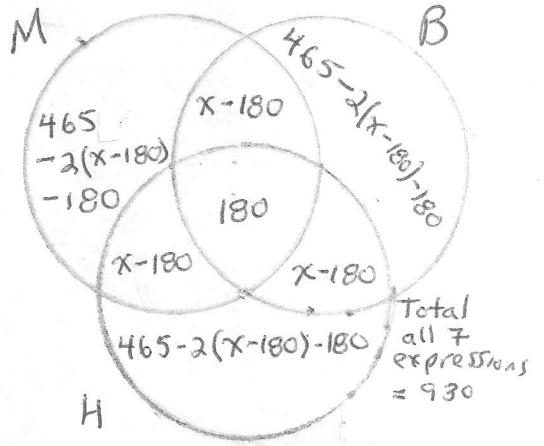
$B = \{\text{Students taking Biology}\}$

$H = \{\text{Students taking History}\}$

$|A|$ = the number of elements in set A

$m = |M|$ = the total number of students taking math

$x = |M \cap H|$ = the total number of students taking Math and History



First, there are 930 students at Hilbert High, and half of them take Math because the number who take Math equals the number who don't take Math: $m = 930 - m$, $2m = 930$, $m = 465$

Next, apply the Principle of Inclusion-Exclusion for three sets:

$$|M \cup B \cup H| = |M| + |B| + |H| - |M \cap B| - |M \cap H| - |B \cap H| + |M \cap B \cap H|$$

From the problem definition, insert the following:

$|M \cup B \cup H| = 930$ (every student takes at least one of the three)

$|M| = |B| = |H| = 465$ (The same number of student take M, B, H)

$x = |M \cap H| = |M \cap B| = |B \cap H|$, and $|M \cap B \cap H| = 180$

$930 = 3(465) - 3x + 180$; $3x = 1395 + 180 - 930 = 645$; $\boxed{x = 215}$

Alternately, arrive at the above equation from the Venn Diagram.

9. First, note the four possibilities for the arguments of $|\cdot|$. (absolute value)

a) $x-3>0,\ x+4>0 \Rightarrow (x-3)(x+4)=8$

b) $x-3>0,\ x+4<0$ not possible

c) $x-3<0,\ x+4>0 \Rightarrow (3-x)(x+4)=8$ because $|x-3|=-(x-3)$

d) $x-3<0,\ x+4<0 \Rightarrow (3-x)(-x-4)=8$

The second condition b) is impossible. The first and last conditions result in the same equation. Therefore there are two cases: a) and c).

a) $(x-3)(x+4)=8;\ x^2+x-12=8;\ x^2+x-20=0;\ (x+5)(x-4)=0$

$x=4$ or $x=5$.

c) $(3-x)(x+4)=8;\ -x^2-x+12=8;\ x^2+x-4=0$

For case c), the quadratic is <u>not</u> factorable, so apply the quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ where $b=1, a=1, c=-4$:

$$x=\frac{-1\pm\sqrt{1^2-4(-4)}}{2}=\frac{-1\pm\sqrt{1+16}}{2}=\frac{-1\pm\sqrt{17}}{2}$$

Combining the solutions to a) and c) and writing the solution in set notation:

$$\boxed{x\in\left\{-5,4,\frac{-1\pm\sqrt{17}}{2}\right\}}$$